

Contiguous relations and summation and transformation formulas for basic hypergeometric series

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Abstract. By using contiguous relations for basic hypergeometric series, we give simple proofs of Bailey's ${}_4\phi_3$ summation, Carlitz's ${}_5\phi_4$ summation, Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation, Sears' ${}_4\phi_3$ transformations, Chen's bibasic summation, Gasper's split poised ${}_{10}\phi_9$ transformation, Chu's bibasic symmetric transformation. Along the same line, finite forms of Sylvester's identity, Jacobi's triple product identity, and Kang's identity are also obtained.

Keywords: q -shifted factorial; basic hypergeometric series; contiguous relations; Sylvester's identity

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1 Introduction

In the previous work [15–17], many terminating summation and transformation formulas for basic hypergeometric series are proved by using contiguous relations and mathematical induction. The present paper is a complement to [15–17]. The difference here is that we will usually apply mathematical induction twice rather than once. Recall that the q -shifted factorial is defined by

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad \text{for } n \in \mathbb{Z}.$$

We employ the abbreviated notation

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad \text{for } n = \infty \quad \text{or} \quad n \in \mathbb{Z}.$$

The *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, b_2, \dots, b_r; q)_k}.$$

An ${}_{r+1}\phi_r$ series is called *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_{r+1} b_r$.

Let

$$F_k(a_1, a_2, \dots, a_{r+1}; q, z) := \frac{(a_1, \dots, a_{r+1}; q)_k}{(q, a_1 q/a_2, \dots, a_1 q/a_{r+1}; q)_k} z^k$$

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be the k -th term of a well-poised hypergeometric series. In [17], the following contiguous relations were established:

$$\begin{aligned} F_k(a_1, a_2, \dots, a_r q, a_{r+1}; q, z) - F_k(a_1, a_2, \dots, a_r, a_{r+1} q; q, z) \\ = \alpha F_{k-1}(a_1 q^2, a_2 q, \dots, a_{r+1} q; q, z), \end{aligned} \quad (1.1)$$

$$\begin{aligned} F_k(a_1, a_2, \dots, a_r, a_{r+1}; q, qz) - F_k(a_1, a_2, \dots, a_r, a_{r+1} q; q, z) \\ = \beta F_{k-1}(a_1 q^2, a_2 q, \dots, a_{r+1} q; q, z), \end{aligned} \quad (1.2)$$

where

$$\begin{aligned} \alpha &= \frac{(a_r - a_{r+1})(1 - a_1/a_r a_{r+1})(1 - a_1)(1 - a_1 q)(1 - a_2) \cdots (1 - a_{r-1})z}{(1 - a_1/a_r)(1 - a_1/a_{r+1})(1 - a_1 q/a_2) \cdots (1 - a_1 q/a_{r+1})}, \\ \beta &= -\frac{(1 - a_1)(1 - a_1 q)(1 - a_2) \cdots (1 - a_r)z}{(1 - a_1/a_{r+1})(1 - a_1 q/a_2) \cdots (1 - a_1 q/a_{r+1})}, \end{aligned}$$

and quite a few terminating summation and transformation formulas for basic hypergeometric series were proved by applying (1.1) or (1.2). Note that Krattenthaler [22] has indicated how to derive contiguous relations from special cases of known formulas (available at <http://www.mat.univie.ac.at/~kratt/papers.html>).

In this paper we shall prove more such identities based on the contiguous relations (1.1), (1.2) or others. Suppose that

$$\sum_{k=0}^n F_{n,k}(a_1, \dots, a_s) = S_n(a_1, \dots, a_s), \quad (1.3)$$

where $F_{n,k}(a_1, \dots, a_s) = 0$ if $k < 0$ or $k > n$. If one can show that the summand $F_{n,k}(a_1, \dots, a_s)$ satisfies the following recurrence relation:

$$F_{n,k}(a_1 q, a_2, \dots, a_s) - F_{n,k}(a_1, a_2, \dots, a_s) = \gamma_n F_{n-v,k-1}(b_1, \dots, b_s), \quad v = 1 \text{ or } 2 \quad (1.4)$$

for some parameters b_1, \dots, b_s , where γ_n is independent of k , then we can prove (1.3) by induction on n . Note that (1.5) is not a special case of Sister Celine's method [26, p. 58, (4.3.1)], as mentioned by [17].

Of course, we need to check that $S_n(a_1, \dots, a_s)$ satisfies the following recurrence relation

$$S_n(a_1 q, a_2, \dots, a_s) - S_n(a_1, a_2, \dots, a_s) = \gamma_n S_{n-v}(b_1, \dots, b_s). \quad (1.5)$$

If $S_n(a_1, \dots, a_s)$ appears as a *closed form* as in Bailey's ${}_4\phi_3$ summation formula (2.1), then the verification of (1.5) is quite easy. If $S_n(a_1, \dots, a_s)$ is of the following form:

$$S_n(a_1, \dots, a_s) = \sum_{k=0}^n G_{n,k}(a_1, \dots, a_s),$$

where $G_{n,k}(a_1, \dots, a_s) = 0$ if $k < 0$ or $k > n$, then we may try to apply q -Gosper's algorithm [21, p. 75] to find a sequence $H_{n,k}(a_1, \dots, a_s)$ of closed forms such that

$$G_{n,k}(a_1 q, a_2, \dots, a_s) - G_{n,k}(a_1, a_2, \dots, a_s) - \gamma_n G_{n-v,k-1}(b_1, \dots, b_s) = H_{n,k} - H_{n,k-1}, \quad (1.6)$$

where $H_{n,n} = H_{n,-1} = 0$. If (1.6) exists, then by telescoping we get (1.5).

It is easy to see that almost all identities are trivial for $n = 0$ or $n = 1$. For the induction step, we need firstly to prove that (1.3) is true for some special a_1 , and secondly give (1.4) (or, in addition, (1.6)). In this way, we shall give simple proofs of Bailey's ${}_4\phi_3$ summation, Carlitz's ${}_5\phi_4$ summation, Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation, Sears' ${}_4\phi_3$ transformations, Chen's bibasic summation, Gasper's split poised ${}_{10}\phi_9$ transformation, Chu's bibasic symmetric transformation. Moreover, we will give a finite form of Sylvester's identity, as well as a finite form of Kang's identity [19].

Note that all terminating identities can be also proved automatically by using the q -Zeilberger algorithm (see, for example, [4, 21, 26]). Moreover, the so-called "Abel's method" (see, for example, [9, 10]) is also a nice method to deal with such identities. Gasper [12] has already developed elementary proofs for many summation formulas for basic hypergeometric series by considering contiguous relations. However, our method is a little different from Gasper's method, since the latter is mainly applied to prove nonterminating basic hypergeometric series, while our method is usually applied to prove terminating series.

2 Bailey's ${}_4\phi_3$ summation formula

Bailey [3] obtained the following ${}_4\phi_3$ summation formula, which was later generalized by Carlitz [5].

Theorem 2.1 (Bailey's ${}_4\phi_3$ summation formula). *For $n \geq 0$, there holds*

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, b, c, -q^{1-n}/bc \\ q^{1-n}/b, q^{1-n}/c, -bc \end{matrix} ; q, q \right] = \begin{cases} \frac{(q, b^2, c^2; q^2)_m (bc; q)_{2m}}{(b, c; q)_{2m} (b^2 c^2; q^2)_m}, & \text{if } n = 2m, \\ 0, & \text{if } n = 2m + 1. \end{cases} \quad (2.1)$$

The following lemma is the $c = q$ case of Theorem 2.1.

Lemma 2.2. *For $n \geq 0$, there holds*

$$\sum_{k=0}^n (-1)^k \frac{(b; q)_k (b; q)_{n-k}}{(-b; q)_{n-k+1} (-bq; q)_k} = \begin{cases} \frac{(b; q)_{n+1}}{(-bq; q)_n (1 - b^2 q^n)}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (2.2)$$

Proof. Since the k -th term and $(n-k)$ -th term only differ by a factor $(-1)^n$, the left-hand side of (2.2) is equal to 0 if n is odd.

Now consider the case where n is even. Since

$$(1 - b^2 q^n) = (1 - b q^{n-k}) - (1 - b q^k) + (1 + b q^{n-k})(1 - b q^k),$$

we have

$$\begin{aligned} (1 - b^2 q^n) \sum_{k=0}^n (-1)^k \frac{(b; q)_k (b; q)_{n-k}}{(-b; q)_{n-k+1} (-bq; q)_k} \\ = \sum_{k=0}^n (-1)^k \frac{(b; q)_k (b; q)_{n+1-k} - (b; q)_{k+1} (b; q)_{n-k}}{(-b; q)_{n-k+1} (-bq; q)_k} + \sum_{k=0}^n (-1)^k \frac{(b; q)_{k+1} (b; q)_{n-k}}{(-b; q)_{n-k} (-bq; q)_k}. \end{aligned} \quad (2.3)$$

Noticing that in the first summation of (2.3) the k -th term and the $(n-k)$ -th term only differ by a sign, we immediately get

$$\sum_{k=0}^n (-1)^k \frac{(b; q)_k (b; q)_{n+1-k} - (b; q)_{k+1} (b; q)_{n-k}}{(-b; q)_{n-k+1} (-bq; q)_k} = 0. \quad (2.4)$$

For the same reason, we have

$$\sum_{k=0}^{n-1} (-1)^k \frac{(b; q)_{k+1} (b; q)_{n-k}}{(-b; q)_{n-k} (-b; q)_{k+1}} = 0. \quad (2.5)$$

The proof then follows from combining (2.3)–(2.5). ■

Proof of Theorem 2.1. It suffices to prove it for the cases $c = q^M$ ($M \geq 1$). We proceed by induction on M . For $M = 1$, Equation (2.1) reduces to (2.2).

Assume that (2.1) holds for $c = q^M$. Let

$$F_{n,k}(b, c, q) = \frac{(q^{-n}; q)_k (b; q)_k (c; q)_k (-q^{1-n}/bc; q)_k q^k}{(q; q)_k (q^{1-n}/b; q)_k (q^{1-n}/c; q)_k (-bc; q)_k}.$$

Applying the contiguous relation (1.1) with $a_r = b$ and $a_{r+1} = -q^{-n}/bc$, we have

$$F_{n,k}(bq, c, q) - F_{n,k}(b, c, q) = \alpha_n F_{n-2,k-1}(bq, cq, q), \quad (2.6)$$

where

$$\alpha_n = \frac{(b + q^{-n}/bc)(1 - c^2)(1 - q^{-n})(1 - q^{1-n})q}{(1 - q^{-n}/b)(1 - q^{1-n}/b)(1 - q^{1-n}/c)(1 + bc)(1 + bcq)}.$$

Let

$$S_n(b, c, q) := \sum_{k=0}^n F_{n,k}(b, c, q).$$

Then summing (2.6) over k from 0 to n gives

$$S_{n-2}(bq, cq, q) = \alpha_n^{-1}(S_n(bq, c, q) - S_n(b, c, q)),$$

from which one can readily check that (2.1) holds for $cq = q^{M+1}$. ■

3 Carlitz's ${}_5\phi_4$ summation formula

By using the q -Pfaff-Saalschütz formula, Carlitz [5] obtained the following ${}_5\phi_4$ summation formula, which is a generalization of [20, Eq. (1)]. Simple proofs of Carlitz's formula have already been given by Guo [15] and Chu and Jia [10]. Here we give a simpler one.

Theorem 3.1 (Carlitz's ${}_5\phi_4$ summation formula). *For $n \geq 0$, there holds*

$${}_5\phi_4 \left[\begin{matrix} q^{-n}, b, c, d, e \\ q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, q^{1-n}/e \end{matrix} ; q, q \right] \\ = q^{m(1+m-n)} (de)^{-m} \frac{(q^{-m}; q)_{2m} (q^{1-n}/bc, q^{1-n}/bd, q^{1-n}/be; q)_m (q^{2m-n}; q)_{n-2m}}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, q^{1-n}/e, q^{n-m}c; q)_m}, \quad (3.1)$$

where $bcde = q^{1+m-2n}$ and $m = \lfloor n/2 \rfloor$.

Proof. It suffices to prove the cases $d = -q^{-M}$ ($M \geq \lfloor n/2 \rfloor$). We proceed by induction on M . For $M = \lfloor n/2 \rfloor$, the identity (3.1) reduces to (2.1). Assume that (3.1) holds for $d = -q^{-M}$.

Let

$$F_{n,k}(b, c, d, q) = \frac{(q^{-n}, b, c, d, q^{1-\lfloor (3n+1)/2 \rfloor} / bcd; q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, q^{\lfloor (n+1)/2 \rfloor} bcd; q)_k} q^k.$$

Applying the contiguous relation (1.1), we have

$$F_{n,k}(b, c, dq, q) - F_{n,k}(b, c, d, q) = \alpha_n F_{n-2,k-1}(bq, cq, dq, q), \quad (3.2)$$

where

$$\alpha_n = - \frac{d(1-b)(1-c)(1-q^{\lfloor (3n+1)/2 \rfloor} bcd^2)(1-q^{\lfloor (n+1)/2 \rfloor} bc)(1-q^n)(1-q^{n-1})q^{\lfloor n/2 \rfloor - 1}}{(1-q^{\lfloor (n+1)/2 \rfloor} bcd)(1-q^{\lfloor (n+3)/2 \rfloor} bcd)(1-q^{n-1}b)(1-q^{n-1}c)(1-q^{n-1}d)(1-q^nd)}.$$

Let

$$S_n(b, c, d, q) := \sum_{k=0}^n F_{n,k}(b, c, d, q).$$

Then summing (3.2) over k from 0 to n gives

$$S_n(b, c, d, q) = S_n(b, c, dq, q) - \alpha_n S_{n-2}(bq, cq, dq, q),$$

from which one can verify that (3.1) holds for $d = -q^{-M-1}$. ■

4 Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation

The following transformation was first obtained by Sears [27, (4.1)]. See also Carlitz [5, (2.4)].

Theorem 4.1 (Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation). *For $a = q^{-n}$, $n = 0, 1, 2, \dots$, there holds*

$${}_3\phi_2 \left[\begin{matrix} a, b, c \\ aq/b, aq/c \end{matrix} ; q, \frac{aqx}{bc} \right] = \frac{(ax; q)_\infty}{(x; q)_\infty} {}_5\phi_4 \left[\begin{matrix} a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, aq/bc \\ aq/b, aq/c, ax, q/x \end{matrix} ; q, q \right]. \quad (4.1)$$

Proof. Let

$$F_{n,k}(x, b, c, q) = \frac{(q^{-n}, b, c; q)_k}{(q^{1-n}/b, q^{1-n}/c, q; q)_k} \left(\frac{q^{1-n}x}{bc} \right)^k.$$

Applying the contiguous relation (1.2) with $a_{r+1} = c$ and $z = xq^{-n}/bc$, we have

$$F_{n,k}(x, b, c, q) - F_{n,k}(x, b, cq, q) = \beta_n F_{n-2,k-1}(xq^{-1}, bq, cq, q),$$

where

$$\beta_n = -\frac{(1 - q^{-n})(1 - q^{1-n})(1 - b)xq^{-n}/bc}{(1 - q^{-n}/c)(1 - q^{1-n}/b)(1 - q^{1-n}/c)}.$$

Let

$$G_{n,k}(x, b, c, q) = \frac{(q^{-n}x; q)_n (q^{-n}; q)_{2k} (q^{1-n}/bc; q)_k q^k}{(q, q^{1-n}/b, q^{1-n}/c, q^{-n}x, q/x; q)_k}.$$

It is easy to verify that

$$G_{n,k}(x, b, c, q) - G_{n,k}(x, b, cq, q) - \beta_n G_{n-2,k-1}(xq^{-1}, bq, cq, q) = 0.$$

On the other hand, for $c = q^{1-n}/b$, the identity (4.1) reduces to the terminating q -binomial theorem (see, for example, [1, (3.3.6)]).

5 Sears' ${}_4\phi_3$ transformations

Sears [28] derived the following two transformations of ${}_4\phi_3$ series. The first one is a q -analogue of Whipple's formula [29]. It is also one of the fundamental formulas in the theory of the basic hypergeometric series. There are several different proofs in the literature. See Andrews and Bowman [2], Ismail [18], Liu [24] and Fang [11]. Here we shall give another simple proof of Sears' ${}_4\phi_3$ transformations.

Theorem 5.1 (Sears' ${}_4\phi_3$ transformations). *If $n \geq 0$, there holds*

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix} ; q, q \right] \\ &= \frac{(e/a, f/a; q)_n}{(e, f; q)_n} a^n {}_4\phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, aq^{1-n}/f \end{matrix} ; q, q \right] \end{aligned} \quad (5.1)$$

$$= \frac{(a, ef/ab, ef/ac; q)_n}{(e, f, ef/abc; q)_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, e/a, f/a, ef/abc \\ ef/ab, ef/ac, q^{1-n}/a \end{matrix} ; q, q \right], \quad (5.2)$$

where $def = abcq^{1-n}$.

Proof. Let

$$F_{n,k}(a, b, d, e, f, q) = \frac{(q^{-n}, a, b, def/abq^{1-n}; q)_k}{(q, d, e, f; q)_k} q^k$$

be the k -th term in the left-hand side of (5.1). One may check that

$$F_{n,k}(aq, b, d, e, f, q) - F_{n,k}(a, b, d, e, f, q) = \alpha_n F_{n-1,k-1}(aq, bq, dq, eq, fq, q), \quad (5.3)$$

where

$$\alpha_n = \frac{(1 - q^{-n})(1 - b)(a - defq^{n-2}/ab)q}{(1 - d)(1 - e)(1 - f)}.$$

Let

$$G_{n,k}(a, b, d, e, f, q) = \frac{(e/a, f/a; q)_n a^n}{(e, f; q)_n} \frac{(q^{-n}, a, d/b, abq^{1-n}/ef; q)_k}{(q, d, aq^{1-n}/e, aq^{1-n}/f; q)_k} q^k.$$

It is easy to verify that

$$\begin{aligned} G_{n,k}(aq, b, d, e, f, q) - G_{n,k}(a, b, d, e, f, q) - \alpha_n G_{n-1,k-1}(aq, bq, dq, eq, fq, q) \\ = H_{n,k} - H_{n,k-1}, \end{aligned} \quad (5.4)$$

where

$$H_{n,k} = \frac{a^n(a^2q^{k-n+2}/ef - 1)(e/a, f/a; q)_n (q^{-n}; q)_{k+1}(aq, d/b, abq^{2-n}/ef; q)_k}{(e, f; q)_n (q, d; q)_k (aq^{1-n}/e, aq^{1-n}/f; q)_{k+1}}.$$

It follows from (5.3) and (5.4) that (5.1) is true for all $a = q^{-M}$ ($M \geq 0$).

To prove (5.2), let

$$P_{n,k}(a, b, d, e, f, q) = \frac{(a, ef/ab, bq^{1-n}/d; q)_n}{(e, f, q^{1-n}/d; q)_n} \frac{(q^{-n}, e/a, f/a, q^{1-n}/d; q)_k}{(q, ef/ab, bq^{1-n}/d, q^{1-n}/a; q)_k}$$

Then we have

$$\begin{aligned} P_{n,k}(aq, b, d, e, f, q) - P_{n,k}(a, b, d, e, f, q) - \alpha_n P_{n-1,k-1}(aq, bq, dq, eq, fq, q) \\ = Q_{n,k} - Q_{n,k-1}, \end{aligned} \quad (5.5)$$

where

$$Q_{n,k} = \frac{q^n(efq^{k-1} - a^2b)(aq, ef/ab; q)_{n-1}(bq^{1-n}/d; q)_n (q^{-n}; q)_{k+1}(e/a, f/a, q^{1-n}/d; q)_k}{ab(e, f, q^{1-n}/d; q)_n (q, ef/ab, bq^{1-n}/d, q^{1-n}/a; q)_k}.$$

It follows from (5.3) and (5.5) that (5.2) is true for all $a = q^{-M}$ ($M \geq 0$). ■

6 Chen's bibasic summation formula

Chu [9] established the following bibasic summation formula:

$$\begin{aligned}
& \sum_{k=-m}^n \frac{(1 - \alpha a_k b_k)(b_k - a_k/\alpha d)}{(1 - \alpha a_0 b_0)(b_0 - a_0/\alpha d)} \frac{\prod_{j=0}^{k-1} (1 - a_j)(1 - a_j/d)(1 - cb_j)(1 - \alpha^2 db_j/c)}{\prod_{j=1}^k (1 - \alpha b_j)(1 - \alpha a_j/c)(1 - \alpha db_j)(1 - ca_j/\alpha d)} \\
&= \frac{(1 - a_0)(1 - a_0/d)(1 - cb_0)(1 - \alpha^2 db_0/c)}{\alpha(1 - \alpha a_0 b_0)(1 - c/\alpha)(b_0 - a_0/\alpha d)(1 - \alpha d/c)} \\
&\quad \cdot \left(\prod_{j=1}^n \frac{(1 - a_j)(1 - a_j/d)(1 - cb_j)(1 - \alpha^2 db_j/c)}{(1 - \alpha b_j)(1 - \alpha a_j/c)(1 - \alpha db_j)(1 - ca_j/\alpha d)} \right. \\
&\quad \left. - \prod_{j=-m}^0 \frac{(1 - \alpha b_j)(1 - \alpha a_j/c)(1 - \alpha db_j)(1 - ca_j/\alpha d)}{(1 - a_j)(1 - a_j/d)(1 - cb_j)(1 - \alpha^2 db_j/c)} \right), \tag{6.1}
\end{aligned}$$

which is a generalization of Gasper and Rahman's bibasic summation formula [14, (3.6.13)].

Setting $c = \alpha/B_n$ and $b_k = B_k/\alpha$ for $k = 0, 1, \dots$, Chen [6] noticed that (6.1) can be simplified as

$$F(n) = \sum_{k=0}^n \frac{(1 - a_k B_k)(dB_k - a_k)}{(1 - a_0 B_n)(dB_n - a_0)} \frac{\sum_{j=0}^{k-1} (B_n - B_j)(1 - dB_n B_j)}{\prod_{j=1}^k (1 - B_n a_j)(dB_n - a_j)} G(k), \tag{6.2}$$

where

$$F(n) = \frac{(1 - B_0)(1 - dB_0)}{(1 - B_n)(1 - dB_n)}, \quad G(n) = \frac{\prod_{j=0}^{n-1} (1 - a_j)(d - a_j)}{\prod_{j=1}^n (1 - B_j)(1 - dB_j)}.$$

Then applying Krattenthaler's matrix inverse [23] to (6.2), Chen [6] obtained

$$G(n) = \sum_{k=0}^n \frac{\prod_{j=0}^{n-1} (1 - B_k a_j)(dB_k - a_j)}{\prod_{j=0, j \neq k}^n (1 - dB_k B_j)(B_k - B_j)} F(k). \tag{6.3}$$

In particular, when $a_k = ap^k$ and $B_k = bq^k$, the identity (6.3) reduces to the following bibasic summation formula.

Theorem 6.1 (Chen's bibasic summation formula). *If $n \geq 0$, there holds*

$$\sum_{k=0}^n \frac{(1 - b)(1 - db)(abq^k, a/dbq^k; p)_n (-1)^k q^{\binom{k+1}{2}}}{(1 - bq^k)(1 - dbq^k)(q, db^2q^k; q)_k (q, db^2q^{2k+1}; q)_{n-k}} = \frac{(a, a/d; p)_n}{(bq, dbq; q)_n}. \tag{6.4}$$

Proof. Let

$$F_{n,k}(a, b, d, q) = \frac{(1 - b)(1 - db)(abq^k, a/dbq^k; p)_n (-1)^k q^{\binom{k+1}{2}}}{(1 - bq^k)(1 - dbq^k)(q, db^2q^k; q)_k (q, db^2q^{2k+1}; q)_{n-k}}.$$

Similarly to the contiguous relation (1.1), we have

$$F_{n,k}(a, b, d, q) - \alpha_n F_{n-1,k}(a, b, d, q) = \beta_n F_{n-1,k-1}(a, bq, d, q), \quad (6.5)$$

where

$$\alpha_n = \frac{(1 - abp^{n-1})(1 - ap^{n-1}/bd)}{(1 - q^n)(1 - db^2q^n)},$$

$$\beta_n = \frac{(abq^n p^{n-1} - 1)(bdq^n - ap^{n-1})(1 - bd)(1 - b)}{db(1 - bdq)(1 - bq)(1 - db^2q^n)(1 - q^n)}.$$

The proof then follows from summing (6.5) over k from 0 to n and proceeding by induction on n . \blacksquare

Note that, the $p = q$ case of (6.4) may be rewritten as

$${}_8\phi_7 \left[\begin{matrix} db^2, q\sqrt{db^2}, -q\sqrt{db^2}, b, db, dbq/a, abq^n, q^{-n} \\ \sqrt{db^2}, -\sqrt{db^2}, dbq, bq, ab, dbq^{1-n}/a, db^2q^{1+n} \end{matrix} ; q, q \right] = \frac{(q, a, a/d, db^2q; q)_n}{(ab, bq, dbq, a/db; q)_n},$$

which is a special case of Jackson's ${}_8\phi_7$ summation formula [20] (see [14, (II.22)]).

7 Gasper's split poised ${}_{10}\phi_9$ transformation

In this section, we show that Gasper's split poised ${}_{10}\phi_9$ transformation formula (see [12]) can also be proved by the same method.

Theorem 7.1 (Gasper's split poised ${}_{10}\phi_9$ transformation). *For $n \geq 0$, there holds*

$$\begin{aligned} & {}_{10}\phi_9 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, a/bc, C/Aq^n, 1/BCq^n, B/Aq^n, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, bcq, 1/Cq^n, BC/Aq^n, 1/Bq^n, 1/Aq^n \end{matrix} ; q, q \right] \\ &= \frac{(aq, bq, cq, aq/bc, Aq/B, Aq/C, BCq; q)_n}{(Aq, Bq, Cq, Aq/BC, aq/b, aq/c, bcq; q)_n} \\ & \times {}_{10}\phi_9 \left[\begin{matrix} A, qA^{\frac{1}{2}}, -qA^{\frac{1}{2}}, B, C, A/BC, c/aq^n, 1/bcq^n, b/aq^n, q^{-n} \\ A^{\frac{1}{2}}, -A^{\frac{1}{2}}, Aq/B, Aq/C, BCq, 1/cq^n, bc/aq^n, 1/bq^n, 1/aq^n \end{matrix} ; q, q \right], \quad (7.1) \end{aligned}$$

where $\lambda = a^2q/bcd$.

Proof. Let

$$F_{n,k}(a, b, c, A, B, C, q) = \frac{(a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, a/bc, C/Aq^n, 1/BCq^n, B/Aq^n, q^{-n}; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, bcq, 1/Cq^n, BC/Aq^n, 1/Bq^n, 1/Aq^n; q)_k} q^k.$$

Similarly to the contiguous relation (1.1) with $a_r = c$ and $a_{r+1} = a/bcq$, we have

$$F_{n,k}(a, b, cq, A, B, C, q) - F_{n,k}(a, b, c, A, B, C, q) = \alpha_n F_{n-1,k-1}(aq^2, bq, cq, A, B, C, q), \quad (7.2)$$

where

$$\alpha_n = \frac{(c - a/bcq)(1 - bq)(1 - aq)(1 - aq^2)(1 - b)}{(1 - a/c)(1 - bcq)(1 - aq/b)(1 - aq/c)(1 - bcq^2)} \\ \times \frac{(1 - C/Aq^n)(1 - 1/BCq^n)(1 - B/Aq^n)(1 - q^{-n})q}{(1 - 1/Cq^n)(1 - BC/Aq^n)(1 - 1/Bq^n)(1 - 1/Aq^n)}.$$

Let

$$G_{n,k}(a, b, c, A, B, C, q) = \frac{(aq, bq, cq, aq/bc, Aq/B, Aq/C, BCq; q)_n}{(Aq, Bq, Cq, Aq/BC, aq/b, aq/c, bcq; q)_n} \\ \times F_{n,k}(A, B, C, a, b, c, q).$$

Then we may verify that

$$G_{n,k}(a, b, cq, A, B, C, q) - G_{n,k}(a, b, c, A, B, C, q) - \alpha_n G_{n-1,k-1}(aq^2, bq, cq, A, B, C, q) \\ = H_{n,k} - H_{n,k-1}, \quad (7.3)$$

where

$$H_{n,k} = \frac{a(1 - b)(1 - q^{n-k})(1 - aq^{n-k+1})(1 - bc^2q/a)}{b(a - c)(1 - aq^{n-k}/bc)(1 - bcq^{n-k+1})(1 - cq)} G_{n,k}(a, b, c, A, B, C, q).$$

Summing (7.2) and (7.3) over k from 0 to n respectively, one sees that both sides of (7.1) satisfy the same recurrence relation

$$S_n(a, b, cq, A, B, C, q) - S_n(a, b, c, A, B, C, q) = \alpha_n S_{n-1}(aq^2, bq, cq, A, B, C, q). \quad (7.4)$$

On the other hand, for the case $c = 1$, Eq. (7.1) reduces to

$$1 = \frac{(q, Aq/B, Aq/C, BCq; q)_n}{(Aq, Bq, Cq, Aq/BC; q)_n} \sum_{k=0}^n \frac{(1 - Aq^{2k})(A, B, C, A/BC; q)_k}{(1 - A)(q, Aq/B, Aq/C, BCq; q)_k} q^k,$$

which is almost trivial since

$$\frac{(1 - Aq^{2k})(A, B, C, A/BC; q)_k}{(1 - A)(q, Aq/B, Aq/C, BCq; q)_k} q^k \\ = \frac{(Aq, Bq, Cq, Aq/BC; q)_k}{(q, Aq/B, Aq/C, BCq; q)_k} - \frac{(Aq, Bq, Cq, Aq/BC; q)_{k-1}}{(q, Aq/B, Aq/C, BCq; q)_{k-1}}.$$

Thus, by (7.4), Eq. (7.1) holds for all $c = q^{-M}$ ($M \geq 0$). This completes the proof. \blacksquare

Chu [9, Corollary 24] has generalized Theorem 7.1 to a bibasic symmetric transformation as follows.

Theorem 7.2 (Chu's bibasic symmetric transformation). *For $n \geq 0$, there holds*

$$\begin{aligned} & \sum_{k=0}^n \frac{(1 - ap^{2k})(a, b, c, a/bc; p)_k (C/Aq^n, 1/BCq^n, B/Aq^n, q^{-n}; q)_k}{(1 - a)(p, ap/b, ap/c, bcp; p)_k (1/Cq^n, BC/Aq^n, 1/Bq^n, 1/Aq^n; q)_k} p^k \\ &= \frac{(ap, bp, cp, ap/bc; p)_n (q, Aq/B, Aq/C, BCq; q)_n}{(p, ap/b, ap/c, bcp; p)_n (Aq, Bq, Cq, Aq/BC; q)_n} \\ & \times \sum_{k=0}^n \frac{(1 - Aq^{2k})(A, B, C, A/BC; q)_k (c/ap^n, 1/bcp^n, b/ap^n, p^{-n}; p)_k}{(1 - A)(q, Aq/B, Aq/C, BCq; q)_k (1/cp^n, bc/ap^n, 1/bp^n, 1/ap^n; p)_k} q^k. \end{aligned} \quad (7.5)$$

We point out that (7.5) can be proved in the same way as (7.1). Let $F_{n,k}(a, b, c, A, B, C; p, q)$ and $G_{n,k}(a, b, c, A, B, C; p, q)$ be the k -th summands in the left-hand side and right-hand side of (7.5) respectively. Then there exist relations for $F_{n,k}$ and $G_{n,k}$ exactly similar to (7.2) and (7.3). For instance, we have

$$F_{n,k}(a, b, cp, A, B, C; p, q) - F_{n,k}(a, b, c, A, B, C; p, q) = \alpha_n F_{n-1,k-1}(ap^2, bp, cp, A, B, C; p, q),$$

where

$$\begin{aligned} \alpha_n &= \frac{(c - a/bcp)(1 - bp)(1 - ap)(1 - ap^2)(1 - b)}{(1 - a/c)(1 - bcp)(1 - ap/b)(1 - ap/c)(1 - bcp^2)} \\ & \times \frac{(1 - C/Aq^n)(1 - 1/BCq^n)(1 - B/Aq^n)(1 - q^{-n})p}{(1 - 1/Cq^n)(1 - BC/Aq^n)(1 - 1/Bq^n)(1 - 1/Aq^n)}. \end{aligned}$$

8 A finite form of Sylvester's identity and Jacobi's triple product identity

From now on, we assume that $|q| < 1$. In the same vein of (1.4), we will prove a finite form of Sylvester's identity and Jacobi's triple product identity. Recall that the q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad n, k \in \mathbb{Z}.$$

Theorem 8.1 (A finite form of Sylvester's and Jacobi's identities). *For $n \geq 0$, there holds*

$$\sum_{k=0}^n (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1 - xq^{2k+1})}{(xq^{k+1}; q)_{n+1}} = 1. \quad (8.1)$$

Proof. Let

$$F_{n,k}(x, q) = (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1 - xq^{2k+1})}{(xq^{k+1}; q)_{n+1}}.$$

Noticing the trivial relation

$$\begin{bmatrix} n \\ k \end{bmatrix} (1 - xq^{n+1}) = \begin{bmatrix} n-1 \\ k \end{bmatrix} (1 - xq^{n+k+1}) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} (1 - xq^{k+1})q^{n-k}, \quad (8.2)$$

we have

$$F_{n,k}(x, q) = \frac{1}{1 - xq^{n+1}} F_{n-1,k}(x, q) - \frac{xq^{n+1}}{1 - xq^{n+1}} F_{n-1,k-1}(xq^2, q). \quad \blacksquare$$

Letting $n \rightarrow \infty$ in (8.1), we immediately obtain Sylvester's identity [1, (9.2.3)]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k q^{k(3k+1)/2} (1 - xq^{2k+1})}{(q; q)_k (xq^{k+1}; q)_{\infty}} = 1. \quad (8.3)$$

On the other hand, if we make the substitutions $n \rightarrow m+n$, $x \rightarrow -xq^{-2m}$ and $k \rightarrow m+k$ in (8.1), and apply the following relation

$$\frac{(-xq^{-2m})^{m+k} q^{3(m+k)^2+m+k)/2}}{(xq^{k-m+1}; q)_{m+n+1}} = \frac{x^{2k} q^{2k^2+k}}{(1/x; q)_{m-k} (xq; q)_{n+k+1}}, \quad (8.4)$$

we obtain

$$\sum_{k=-m}^n \begin{bmatrix} m+n \\ m+k \end{bmatrix} \frac{(1 - xq^{2k+1}) x^{2k} q^{2k^2+k}}{(1/x; q)_{m-k} (xq; q)_{n+k+1}} = 1. \quad (8.5)$$

Letting $m, n \rightarrow \infty$ in (8.5), we immediately get

$$\sum_{k=-\infty}^{\infty} (1 - xq^{2k+1}) x^{2k} q^{2k^2+k} = (1/x, xq, q; q)_{\infty}, \quad (8.6)$$

which is Jacobi's triple product identity (see [1, p. 21]).

Note that finite forms of the quintuple product identity (see [14, p. 147])

$$\sum_{k=-\infty}^{\infty} (z^2 q^{2k+1} - 1) z^{3k+1} q^{k(3k+1)/2} = (q, z, q/z; q)_{\infty} (qz^2, q/z^2; q^2)_{\infty},$$

which is related to Jacobi's triple product identity, were given by [7, 17, 25].

9 A finite form of Kang's identity

In this section, we give a finite form of Kang's [19] generalization of Sylvester's identity. Similarly to the previous section, we only give the corresponding contiguous relation for the k th terms in our proof.

Theorem 9.1. For $n \geq 0$, there holds

$$\sum_{k=0}^n (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(cq^{-k}, dq^{-k}; q)_k (1 - xq^{2k+1})}{(cx, dx; q)_{k+1} (xq^{k+1}; q)_{n+1}} = \frac{(cdx; q)_n}{(cx, dx; q)_{n+1}}. \quad (9.1)$$

Proof. Let

$$F_{n,k}(c, d, x, q) = (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(cq^{-k}, dq^{-k}; q)_k (1 - xq^{2k+1})}{(cx, dx; q)_{k+1} (xq^{k+1}; q)_{n+1}}.$$

Then

$$\begin{aligned} F_{n,k}(c, d, x, q) &= \frac{1}{1 - xq^{n+1}} F_{n-1,k}(c, d, x, q) \\ &\quad - \frac{(1 - cq^{-1})(1 - dq^{-1})xq^{n+1}}{(1 - cx)(1 - dx)(1 - xq^{n+1})} F_{n-1,k-1}(cq^{-1}, dq^{-1}, xq^2, q). \end{aligned}$$

■

Letting $n \rightarrow \infty$ in (9.1), we obtain the following identity due to Kang [19, (6.9)]:

$$\sum_{k=0}^{\infty} (-1)^k x^k q^{k(3k+1)/2} \frac{(aq^{-k}, bq^{-k}, xq; q)_k (1 - xq^{2k+1})}{(q; q)_k (ax, bx; q)_{k+1}} = \frac{(abx, xq; q)_{\infty}}{(ax, bx; q)_{\infty}}$$

for $|ax| < 1$ and $|bx| < 1$.

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